Int. J. Solids Structures Vol. 30, No. 12, pp. 1565–1577, 1993
Printed in Great Britain
0020–7683/93 $6.00+.00

BOUNDS FOR THE MAXIMUM DEFLECTION AND CURVATURE OF NONUNIFORM BEAM-COLUMNS

ROBERT PICHÉ
Department of Mathematics, Tampere University of Technology, P.O. Box 692, 33101 Tampere, Finland

(*Received* 13 *October* 1992; *in revised form* 10 *January* 1993)

Abstract—An upper bound formula is derived for the maximum deflection and curvature of nonuniform linear elastic beam-columns subject to both transverse and axial loading. The formulae are derived using the norms of the operators appearing in the integral equation for the beam-column. Numerical values of formula parameters are given for six elementary sets of boundary conditions.

1. INTRODUCTION

A beam subject to axial compression in addition to transverse loads cannot be analysed by simple superposition because the axial load causes a feedback-type interaction between the bending moment and the deflection. For uniform beam-columns, exact solutions such as the well-known "secant formula" have been worked out and tabulated in Roark and Young (1975) for various loadings and boundary conditions. Systematic methods for exact analysis of stepped beam-columns made up of uniform sections have been developed by Lee and Kuo (1991) using integral equations. Approximate formulae such as Timoshenko's "amplification factor" (Timoshenko and Gere, 1961) are also widely used. The amplification factor is of the form $1/(1-\lambda/\lambda_{\text{crit}})$, where λ is the axial load magnitude and λ_{crit} is the critical buckling load. This amplification factor appears in the upper-bound estimate derived by Flavin (1976) for beam-column deflections.

In this work, the beam-column is analysed using integral operator theory. It has long been recognized that a basic inequality associated with the Hilbert–Schmidt norm of integral equation operators leads to a useful and accurate lower-bound estimate of the critical load in buckling problems for linear elastic structures (Baruch, 1973; Mikhlin, 1964). Recently Piché (1993) has investigated the use of other norms for computing lower-bound estimates of the critical load for discrete structures. In the present work, the infinity norm is used to derive an upper bound estimate for beam-column deflections and curvatures.

The basic norm inequality which is the basis for this work is presented in Section 2. Explicit formulae for the integral operators for the six elementary sets of beam boundary conditions are derived in Section 3. Numerical results and comparisons with other upper-bound formulae are given in Section 4.

2. BASIC OPERATOR NORM INEQUALITY

As will be shown in the next section, the equation for the deflection $y(x)$ of a linear elastic beam-column with axial load parameter λ and standard boundary conditions can be written in the form

$$y(x) = y_0(x)+\lambda\int_0^l F(x,\xi)y(\xi)\,\mathrm{d}\xi, \tag{1}$$

where $y_0(x)$ is the deflection due to lateral loads when no axial load is present, and the function $F(x,\xi)$ is known as the kernel of the integral equation. An integral equation of the form (1) is known as a Fredholm equation of the second kind.

The integral equation (1) may be written in the symbolic form

$$y = y_0 + \lambda \mathscr{F} y,$$

where $\mathscr{F}$ represents the integral operator acting on the function y. This symbolic equation emphasizes the positive feedback nature of the system: The deflection function y is equal to the lateral loading term y_0 plus the term $\lambda \mathscr{F} y$ formed by a linear operator on the deflection itself. When the axial load term λ is positive, this feedback effect tends to amplify the deflection. The concept of "operator norm" gives us a measure of this amplification effect.

A real function $\|\cdot\|$ is called a norm for an integral operator if it satisfies the usual axioms for a norm and, in addition, the submultiplicative property

$$\|\mathscr{F}\mathscr{G}\| \leqslant \|\mathscr{F}\| \, \|\mathscr{G}\|$$

for any pair of integral operators $\mathscr{F}$ and $\mathscr{G}$. Here the notation $\mathscr{F}\mathscr{G}$ means the composition operator:

$$(\mathscr{F}\mathscr{G}y)(x) := \int_0^l F(x,\eta)(\mathscr{G}y)(\eta)\,\mathrm{d}\eta = \int_0^l \int_0^l F(x,\eta)G(\eta,\xi)y(\xi)\,\mathrm{d}\eta\,\mathrm{d}\xi.$$

The kernel of the composition operator $\mathscr{F}\mathscr{G}$ is

$$\int_0^l F(x,\eta)G(\eta,\xi)\,\mathrm{d}\eta.$$

An associated function norm can be defined in terms of the operator norm, as follows. If y is a continuous function, and the kernel of the operator $\mathscr{Y}$ is constructed from y according to the formula

$$Y(x,\xi) := y(x),$$

then the formula

$$\|y\| := \|\mathscr{Y}\|$$

defines a function norm for y. Applying this definition to the expression $\mathscr{F}y$ gives the useful inequality

$$\begin{aligned}\|\mathscr{F}y\| &= \left\| \int_0^l F(x,\eta)y(\eta)\,\mathrm{d}\eta \right\| \\ &= \left\| \int_0^l F(x,\eta)Y(\eta,\xi)\,\mathrm{d}\eta \right\| \\ &= \|\mathscr{F}\mathscr{Y}\| \\ &\leqslant \|\mathscr{F}\| \, \|y\|.\end{aligned}$$

Two examples of integral operator norms are the 2-norm

$$\|\mathscr{F}\|_2 = \left(\int_0^l \int_0^l |F(x,\xi)|^2 \,\mathrm{d}\xi\,\mathrm{d}x \right)^{1/2}$$

and the ∞-norm

$$\|\mathscr{F}\|_\infty = \sup_{0\leqslant x\leqslant l} \int_0^l |F(x,\xi)|\,\mathrm{d}\xi. \tag{2}$$

The labels 2 and ∞ applied to the operator norms given above are used because the associated function norm is equal to the usual Lebesgue space $\mathbf{L}_p[0, l]$ norm for $p = 2, \infty$:

$$\|y\|_2 = \left(\int_0^l l|y(x)|^2\,\mathrm{d}x\right)^{1/2}, \quad \|y\|_\infty = \sup_{0\leqslant x\leqslant l} l|y(x)|.$$

The 2-norm is also known as the Hilbert–Schmidt norm, and this is the norm which is most commonly used in the theory of integral equations. For structural engineering purposes, however, it is the ∞-norm which is of most practical use, since structural design constraints are generally in terms of maximum values. In the remaining discussion we will therefore mainly be concerned with the ∞-norm.

A basic inequality relating any operator norm to the set Λ of eigenvalues of the homogeneous equation

$$y = \lambda\mathscr{F}y$$

is

$$1/\|\mathscr{F}\| \leqslant |\lambda| \quad \forall\, \lambda \in \Lambda. \tag{3}$$

The inequality is proved as follows (Horn and Johnson, 1985). For any eigenvalue λ and corresponding eigenvector y we have

$$\|y\| = \|\lambda\mathscr{F}y\| \leqslant |\lambda|\,\|\mathscr{F}\|\,\|y\|$$

from which (3) follows immediately. In the context of beam-column problems, inequality (3) is essentially a lower-bound estimate of the critical buckling load $\lambda_{\mathrm{crit}} := \min_\Lambda |\lambda|$. The lower-bound buckling load formulae presented in Baruch (1973) and Mikhlin (1964) are special cases of (3) using the Hilbert–Schmidt norm.

Applying the basic properties of operator norms outlined above to eqn (1), the upper-bound formula for beam-column deflections can be derived quite rapidly. We have

$$\|y\|_\infty = \|\lambda\mathscr{F}y + y_0\|_\infty \leqslant |\lambda|\,\|\mathscr{F}\|_\infty\,\|y\|_\infty + \|y_0\|_\infty.$$

If λ is a constant satisfying

$$|\lambda| < \|\mathscr{F}\|_\infty$$

then it immediately follows that

$$\|y\|_\infty \leqslant \frac{\|y_0\|_\infty}{1 - |\lambda|\,\|\mathscr{F}\|_\infty}, \tag{4}$$

and this is the proposed bound on the ∞ norm of the deflection y, valid for all $|\lambda| < \|\mathscr{F}\|_\infty$.

The formula (4) is exact for $\lambda = 0$ and has a singularity at $|\lambda| = \|\mathscr{F}\|_\infty$. The formula is similar in form to the "amplification factor" formula of Timoshenko and Gere (1961), but instead of $1/(1-\lambda/\lambda_{\mathrm{crit}})$ we have $1/(1-|\lambda|\,\|\mathscr{F}\|_\infty)$. As a consequence of the inequality (3), the factor in formula (4) is always greater than the Timoshenko–Gere amplification factor.

Similar integral equations and inequalities can be derived for other quantities of interest such as slope, bending moment, or curvature. For the sake of brevity we shall only consider curvature. The integral equation for the (small-deflection approximation to) curvature $y''(x)$ of a linear elastic beam-column with axial load parameter λ has a form similar to (1),

$$y''(x) = y_0''(x) + \lambda \int_0^l H(x, \xi) y''(\xi)\, \mathrm{d}\xi \tag{5}$$

where $y_0''(x)$ is the curvature due to lateral loads when no axial load is present and the function $H(x, \xi)$ is the kernel of the integral equation. Following a similar development as for the deflection equation, an upper bound for the maximum curvature is given by an inequality similar to (4):

$$\| y'' \|_\infty \leqslant \frac{\| y_0'' \|_\infty}{1 - |\lambda|\, \| \mathscr{H} \|_\infty}. \tag{6}$$

The proposed formulae for maximum deflection and curvature have three components: the axial load parameter λ, the maximum deflection or curvature for the beam with no axial load, and the operator norm. The first component is a numerical parameter. The second component requires the solution of a standard beam boundary value problem. This is usually much easier to solve than the beam-column problem, and solutions for a wide range of standard beams are tabulated in the literature (Roark and Young, 1975). The last component is a constant which depends on the boundary conditions and on the distribution of axial load and bending stiffness. In order to determine this constant, it is necessary to find the integral operator kernel. This question is addressed in the following section. Numerical values of norms are tabulated in Section 4, and examples of the use of the formulae are given.

3. DERIVATION OF KERNELS FOR COLUMNS

There are many ways of deriving the integral operator kernel for a beam-column. Two derivations are given here. The first derivation proceeds directly from the standard differential equation boundary value problem. The second derivation brings out the physical significance of the integral operator kernel, thus in principle opening the way to its determination from experimental data.

Consider an elementary beam-column with bending stiffness $EIa(x) > 0$, where EI is constant and $a(x)$ is a dimensionless function which may vary with position x along the span l. The beam-column has constant axial compression $\lambda EI/l^2$, where λ is a dimensionless constant, and lateral load $f(x)$. The differential equation for the equilibrium of the column is given by

$$l^2[a(x)y''(x)]'' + \lambda y''(x) = f(x) \tag{7}$$

together with a set of homogeneous boundary conditions. Let $y_0(x)$ denote the deflection due to lateral loads when no axial load is present. Then $y_0(x)$ satisfies the differential equation

$$l^2[a(x)y_0''(x)]'' = f(x) \tag{8}$$

together with the same boundary conditions as (7). Introducing the function

$$z(x) := y(x) - y_0(x) \tag{9}$$

into (7) and making use of (8) gives

$$l^2[a(x)z''(x)]''+\lambda y''(x) = 0, \tag{10}$$

where $z(x)$ satisfies the same boundary conditions as $y(x)$.

There are six basic column problems, corresponding to the following six sets of homogeneous boundary conditions:

clamped–sliding: $y(0) = y'(0) = y'(l) = (ay'')'(l) = 0$;

pinned–pinned: $y(0) = y''(0) = y(l) = y''(l) = 0$;

pinned–sliding: $y(0) = y''(0) = y'(l) = (ay'')'(l) = 0$;

clamped–free: $y'(0) = y(l) = y''(l) = [l^2(ay'')'+\lambda y'](l) = 0$;

clamped–pinned: $y(0) = y'(0) = y(l) = y''(l) = 0$;

clamped–clamped: $y(0) = y'(0) = y(l) = y'(l) = 0$.

Note that for the clamped–free column, the deflection $y(x)$ is measured relative to a shifted x-axis: The shift is such as to give a zero deflection at the free end of the column. This minor departure from convention ensures that the resulting integral equation has the same form as for the other column problems.

The differential equation (10) together with given boundary conditions may be transformed into an integral equation by successive integration, as follows. Integrating the differential equation (10) over $(0, x)$ gives

$$l^2(a(x)z''(x))' = -\lambda y'(x)+c_1 \tag{11}$$

where c_1 is a constant. A second integration over $(0, x)$ gives

$$l^2a(x)z''(x) = -\lambda y(x)+c_1x+c_2$$

where c_2 is a constant. Dividing through by $l^2a(x)$ gives

$$z''(x) = \frac{-\lambda y(x)+c_1x+c_2}{l^2a(x)}.$$

Another integration over $(0, x)$ gives

$$\begin{aligned} z'(x) &= \int_0^x \frac{-\lambda y(\xi)+c_1\xi+c_2}{l^2a(\xi)}\,\mathrm{d}\xi+c_3 \\ &= -\lambda\int_0^x \frac{y(\xi)}{l^2a(\xi)}\,\mathrm{d}\xi+\frac{c_1A_1(x)+c_2A_0(x)}{l^2}+c_3, \end{aligned}$$

where c_3 is a constant, and the following notation is introduced for the moment integrals of $1/a$:

$$A_k(x) := \int_0^x \frac{\xi^k}{a(\xi)}\,\mathrm{d}\xi \quad (k = 0, 1, 2).$$

A final integration over $(0, x)$ gives

$$z(x) = -\lambda \int_0^x \int_0^\xi \frac{y(\eta)}{l^2 a(\eta)} \, d\eta \, d\xi + \int_0^x \frac{c_1 A_1(\xi) + c_2 A_0(\xi)}{l^2} \, d\xi + c_3 x + c_4$$

$$= -\lambda \int_0^x \frac{(x-\xi)y(\xi)}{l^2 a(\xi)} \, d\xi + c_1 \frac{x A_1(x) - A_2(x)}{l^2} + c_2 \frac{x A_0(x) - A_1(x)}{l^2} + c_3 x + c_4, \quad (12)$$

where c_4 is a constant.

The values of the four constants of integration are found from the boundary conditions. For example, substituting the boundary conditions for the clamped–sliding column into eqns (11)–(12) gives $c_1 = c_3 = c_4 = 0$ and

$$c_2 = \lambda \int_0^l \frac{y(\xi)}{A_0(l) a(\xi)} \, d\xi.$$

Substituting these values of the integration constants into eqn (12) gives

$$z(x) = -\lambda \int_0^x \frac{(x-\xi)y(\xi)}{l^2 a(\xi)} \, d\xi + \lambda \int_0^l y(\xi) \frac{x A_0(x) - A_1(x)}{l^2 A_0(l) a(\xi)} \, d\xi.$$

This integral equation can be written in the form

$$z(x) = \lambda \int_0^x F_1(x,\xi) y(\xi) \, d\xi + \lambda \int_x^l F_2(x,\xi) y(\xi) \, d\xi \quad (13)$$

with

$$F_2(x,\xi) = \frac{x A_0(x) - A_1(x)}{l^2 a(\xi) A_0(l)}$$

and

$$F_1(x,\xi) = F_2(x,\xi) + \frac{\xi - x}{l^2 a(\xi)}. \quad (14)$$

Defining the function $F(x, \xi)$ as

$$F(x,\xi) = \begin{cases} F_1(x,\xi) & \text{for } \xi \leqslant x \\ F_2(x,\xi) & \text{for } x < \xi \end{cases} \quad (15)$$

the integral equation (13) may be written

$$z(x) = \lambda \int_0^l F(x,\xi) y(\xi) \, d\xi.$$

Substituting the formula (9) for $z(x)$ yields

$$y(x) = y_0(x) + \lambda \int_0^l F(x,\xi) y(\xi) \, d\xi \quad (16)$$

which is precisely the integral equation (1) presented at the beginning of the previous section.

The integral equations for the other column problems are derived in a similar fashion and are all in the form (15)–(16). The kernels for the six elementary sets of boundary conditions are as follows:

clamped–sliding: $F_2(x,\xi) = \dfrac{xA_0(x)-A_1(x)}{l^2a(\xi)A_0(l)}$;

pinned–pinned: $F_2(x,\xi) = \dfrac{x(l-\xi)}{l^3a(\xi)}$;

pinned–sliding: $F_2(x,\xi) = \dfrac{x}{l^2a(\xi)}$;

clamped–free: $F_2(x,\xi) = \dfrac{l-\xi}{l^2a(\xi)}$;

clamped–pinned: $F_2(x,\xi) = \dfrac{(l-\xi)[(x+l)A_1(x)-A_2(x)-xlA_0(x)]}{[2lA_1(l)-A_2(l)-l^2A_0(l)]a(\xi)l^2}$;

clamped-clamped: $F_2(x,\xi) = \{[\xi A_0(l)-A_1(l)][xA_1(x)-A_2(x)]$
$+[A_2(l)-\xi A_1(l)][xA_0(x)-A_1(x)]\}\{[A_0(l)A_2(l)-A_1(l)A_1(l)]l^2a(\xi)\}^{-1}$.

In all six column problems the other section of the deflection equation kernel is given by (14).

For uniform columns ($a \equiv 1$), the above kernels simplify to the following expressions:

clamped–sliding: $F_2(x,\xi) = \dfrac{x^2}{2l^2}$,

$$F_1(x,\xi) = \frac{x^2-2xl+2l\xi}{2l^2};$$

pinned–pinned: $F_2(x,\xi) = \dfrac{x(l-\xi)}{l^3}$,

$$F_1(x,\xi) = \frac{\xi(l-x)}{l^3};$$

pinned–sliding: $F_2(x,\xi) = \dfrac{x}{l^2}$,

$$F_1(x,\xi) = \frac{\xi}{l^2};$$

clamped–free: $F_2(x,\xi) = \dfrac{l-\xi}{l^2}$,

$$F_1(x,\xi) = \frac{l-x}{l^2};$$

clamped–pinned: $F_2(x,\xi) = \dfrac{x^2(\xi-l)(x-3l)}{2l^5}$,

$$F_1(x,\xi) = \frac{x^3\xi-x^3l+3x^2l^2-3x^2\xi l+2\xi l^3-2l^3x}{2l^5};$$

clamped–clamped: $F_2(x,\xi) = \dfrac{x^2(-xl+2x\xi+2l^2-3l\xi)}{l^5}$,

$$F_1(x,\xi) = \frac{-x^3l+2x^3\xi+2x^2l^2-3x^2\xi l+l^3\xi-l^3x}{l^5}.$$

Formulae for deflection equation kernels for uniform columns are also given by Arbabi and Li (1991).

The derivation given above is purely mathematical and lacks a clear physical interpretation. For this reason, we present the following alternative derivation based on the equivalent beam concept introduced by Baruch (1973). In this approach, the term $\lambda y''(x)$ in (7) representing axial force effect is treated as a fictitious external distributed moment load of magnitude $\lambda y'(x)$ per unit length. Applying the Maxwell–Betti reciprocal theorem for the beam deflection yields

$$y(x) = \int_0^l v(x,\xi) f(\xi)\,\mathrm{d}\xi + \lambda \int_0^l \frac{\partial v(x,\xi)}{\partial \xi} y'(\xi)\,\mathrm{d}\xi,$$

where $v(x,\xi)$ is the deflection of the beam at point ξ due to a unit force at point x. Substituting

$$y_0(x) = \int_0^l v(x,\xi) f(\xi)\,\mathrm{d}\xi$$

and integrating by parts gives

$$y(x) = y_0(x) + \lambda \left[y(\xi) \frac{\partial v(x,\xi)}{\partial \xi} \right]_{\xi=0}^{\xi=l} - \lambda \int_0^l \frac{\partial^2 v(x,\xi)}{\partial \xi^2} y(\xi)\,\mathrm{d}\xi. \tag{17}$$

The term in brackets vanishes when the boundary conditions for each of the six standard beam problems are applied, leaving the beam-column equation in the form of (16) with

$$F(x,\xi) = -\frac{\partial^2 v(x,\xi)}{\partial \xi^2}. \tag{18}$$

For this class of boundary conditions, therefore, the kernel has a physical significance: It is the (small-deflection approximation to) negative curvature of the beam at point ξ due to a unit lateral force at point x.

It may be noted that the deflection equation kernels for the uniform pinned–pinned, pinned–sliding, and clamped–free columns are symmetric, but that the kernels for the other columns are not. It may appear surprising that the integral equation kernel is not always symmetric even though the underlying ordinary differential equation boundary value problem is self-adjoint. Baruch (1973) has shown, however, that if the integral equation is written using the slope $y'(z)$ as the unknown function, then the kernel is symmetric for any set of boundary conditions corresponding to a self-adjoint problem. Thus, the lack of symmetry is not due to an error in the derivation, but rather is a consequence of using deflection as the variable in place of slope.

The derivation of the integral equations for curvature, eqn (6), is similar to the derivation presented in the previous paragraphs, and is not given here. For all six standard beam problems considered here, the kernel $H(x,\xi)$ for the curvature equation can be expressed in terms of the kernel for the deflection by the relation

$$H(x,\xi) = F(\xi,x).$$

This can be shown by differentiation and integration by parts of (17).

4. NUMERICAL RESULTS

4.1. *Uniform beam-columns*

The operator norms for uniform beam-columns are calculated using the formula (2) and the exact values are listed in Table 1. The reciprocals of the norms listed in Table 1 are smaller than the critical buckling load for the corresponding column, in agreement with (3).

The following example illustrates the accuracy of the norm-based upper bound formula. Consider a pinned–pinned beam-column with equal and opposite couples M applied at the ends and compressive load $\lambda EI/l^2$. The exact solution given in Timoshenko and Gere (1961) is

$$\|y\|_\infty = \frac{Ml^2}{8EI}\frac{8}{\lambda}(\sec(\sqrt{\lambda}/2)-1) = \|y_0\|_\infty\frac{8}{\lambda}(\sec(\sqrt{\lambda}/2)-1),$$

for the maximum deflection and

$$\|y''\|_\infty = \frac{M}{EI}\sec(\sqrt{\lambda}/2) = \|y_0''\|_\infty \sec(\sqrt{\lambda}/2)$$

for the maximum absolute curvature. The formula of Flavin (1976) gives

$$y_{\text{Flavin}} = \|y_0\|_\infty \frac{\sqrt{4/3}}{1-\lambda/\pi^2}$$

as an upper bound estimate for the maximum deflection and

$$y''_{\text{Flavin}} = \|y_0''\|_\infty\left(1+\frac{\lambda\sqrt{4/192}}{1-\lambda/\pi^2}\right)$$

as an upper bound estimate for the maximum absolute curvature. The norm-based formulae are

$$\|y\|_\infty \leqslant \|y_0\|_\infty \frac{1}{1-\lambda/8}$$

and

$$\|y''\|_\infty \leqslant \|y_0''\|_\infty \frac{1}{1-\lambda/8}.$$

Table 1. Norms for uniform beam-columns

Boundary condition	$\|\mathscr{F}\|_\infty$	$\|\mathscr{H}\|_\infty$	λ_{crit}
Clamped–sliding	1/4	1/3	π^2
Pinned–pinned	1/8	1/8	π^2
Pinned–sliding	1/2	1/2	$\pi^2/4$
Clamped–free	1/2	1/2	$\pi^2/4$
Clamped–pinned	$3/2-\sqrt{2}$	1/8	20.19
Clamped–clamped	1/16	1/12	$4\pi^2$

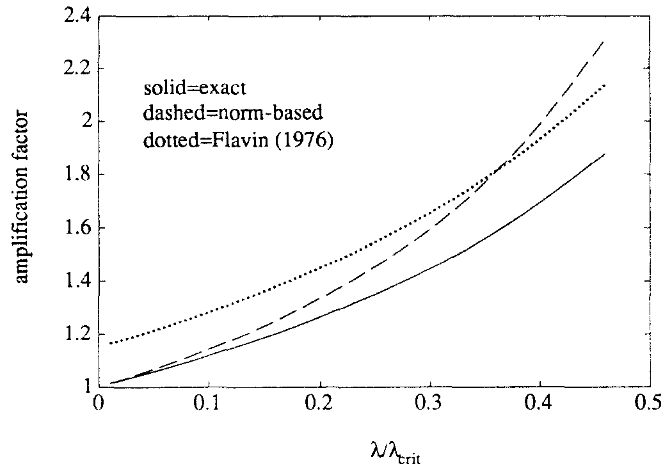

Fig. 1. Maximum deflection in pinned–pinned beam-column with end couples.

The graphs of the formulae for maximum deflection and curvature are plotted in Figs 1 and 2, respectively. In both graphs it may be seen how the norm-based estimate is more accurate for small axial compression, while Flavin's formula tends to be more accurate for axial compression values greater than about 40% of the critical buckling load. The better accuracy for small compressive loads is to be expected, since Flavin's formula generally tends to overestimate in the case $\lambda = 0$ while the norm-based formula is always exact in this case.

4.2. *Tapered beam-columns*

For nonuniform beam-columns, the operator norm formula (2) is difficult to compute analytically, but may be computed using numerical integration and optimization codes. In Tables 2–13, the norm values are tabulated for tapered beams with bending stiffness variation of the form

$$a(x) = (1+kx)^n,$$

with $n = 1, 2, 3, 4$ and k chosen to give different ratios of stiffnesses at the ends. These tables are organized to correspond to the tables in Roark and Young (1975) for beams without axial loads.

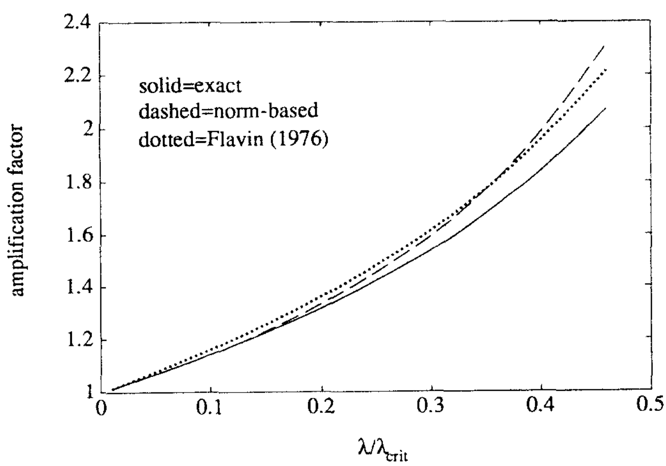

Fig. 2. Maximum curvature in pinned–pinned beam-column with end couples.

Table 2. Norms for deflection of clamped–sliding beam-column with taper $a(x) = (1+kx)^n$

$a(l)/a(0)$	$n=1$	$n=2$	$n=3$	$n=4$
0.25	0.4503	0.4773	0.4869	0.4919
0.5	0.3443	0.3495	0.3512	0.3521
2	0.1721	0.1747	0.1756	0.1760
4	0.1126	0.1193	0.1217	0.1230
8	0.0702	0.0797	0.0833	0.0853

Table 3. Norms for deflection of pinned–pinned beam-column with taper $a(x) = (1+kx)^n$

$a(l)/a(0)$	$n=1$	$n=2$	$n=3$	$n=4$
0.25	0.2167	0.2386	0.2467	0.2508
0.5	0.1704	0.1747	0.1762	0.1769
2	0.0852	0.0874	0.0881	0.0885
4	0.0542	0.0597	0.0617	0.0627
8	0.0323	0.0398	0.0429	0.0445

Table 4. Norms for deflection of pinned–sliding beam-column with taper $a(x) = (1+kx)^n$

$a(l)/a(0)$	$n=1$	$n=2$	$n=3$	$n=4$
0.25	1.1312	1.2274	1.2599	1.2761
0.5	0.7726	0.7885	0.7937	0.7963
2	0.3069	0.3129	0.3150	0.3161
4	0.1793	0.1931	0.1984	0.2012
8	0.1004	0.1176	0.1250	0.1290

Table 5. Norms for deflection of clamped–free beam-column with taper $a(x) = (1+kx)^n$

$a(l)/a(0)$	$n=1$	$n=2$	$n=3$	$n=4$
0.25	0.7172	0.7726	0.7937	0.8047
0.5	0.6137	0.6257	0.6300	0.6321
2	0.3863	0.3942	0.3969	0.3981
4	0.2828	0.3069	0.3150	0.3190
8	0.1966	0.2359	0.2500	0.2571

Table 6. Norms for deflection of clamped–pinned beam-column with taper $a(x) = (1+kx)^n$

$a(l)/a(0)$	$n=1$	$n=2$	$n=3$	$n=4$
0.25	0.4502	0.4772	0.4869	0.4919
0.5	0.3443	0.3494	0.3512	0.3521
2	0.1721	0.1747	0.1756	0.1760
4	0.1126	0.1193	0.1217	0.1230
8	0.0702	0.0797	0.0833	0.0853

Table 7. Norms for deflection of clamped–clamped beam-column with taper $a(x) = (1+kx)^n$

$a(l)/a(0)$	$n=1$	$n=2$	$n=3$	$n=4$
0.25	0.1121	0.1205	0.2174	0.1251
0.5	0.0860	0.0876	0.1514	0.0884
2	0.0430	0.0438	0.0708	0.0442
4	0.0280	0.0301	0.0478	0.0313
8	0.0174	0.0204	0.0320	0.0221

Table 8. Norms for curvature of clamped–sliding beam-column with taper $a(x) = (1+kx)^n$

$a(l)/a(0)$	$n=1$	$n=2$	$n=3$	$n=4$
0.25	0.4627	0.4548	0.4535	0.4531
0.5	0.3607	0.3611	0.3612	0.3612
2	0.3034	0.3036	0.3037	0.3038
4	0.2723	0.2726	0.2731	0.2735
8	0.2420	0.2412	0.2425	0.2433

Table 9. Norms for curvature of pinned–pinned beam-column with taper $a(x) = (1+kx)^n$

$a(l)/a(0)$	$n=1$	$n=2$	$n=3$	$n=4$
0.25	0.2222	0.2500	0.2599	0.2650
0.5	0.1716	0.1768	0.1785	0.1794
2	0.0858	0.0884	0.0893	0.0897
4	0.0556	0.0625	0.0650	0.0662
8	0.0341	0.0442	0.0481	0.0501

Table 10. Norms for curvature of pinned–sliding beam-column with taper $a(x) = (1+kx)^n$

$a(l)/a(0)$	$n=1$	$n=2$	$n=3$	$n=4$
0.25	2.0000	2.0000	2.0000	2.0000
0.5	1.0000	1.0000	1.0000	1.0000
2	0.2679	0.2735	0.2756	0.2767
4	0.1505	0.1667	0.1732	0.1767
8	0.0842	0.1074	0.1174	0.1227

Table 11. Norms for curvature of clamped–free beam-column with taper $a(x) = (1+kx)^n$

$a(l)/a(0)$	$n=1$	$n=2$	$n=3$	$n=4$
0.25	0.6019	0.6667	0.6928	0.7066
0.5	0.5359	0.5469	0.5511	0.5533
2	0.5000	0.5000	0.5000	0.5000
4	0.5000	0.5000	0.5000	0.5000
8	0.5000	0.5000	0.5000	0.5000

Table 12. Norms for curvature of clamped–pinned beam-column with taper $a(x) = (1+kx)^n$

$a(l)/a(0)$	$n=1$	$n=2$	$n=3$	$n=4$
0.25	0.1470	0.1507	0.2504	0.1524
0.5	0.1371	0.1380	0.2186	0.1384
2	0.1114	0.1120	0.1584	0.1124
4	0.0974	0.0993	0.1332	0.1006
8	0.0839	0.0871	0.1095	0.0896

Table 13. Norms for curvature of clamped–clamped beam-column with taper $a(x) = (1+kx)^n$

$a(l)/a(0)$	$n=1$	$n=2$	$n=3$	$n=4$
0.25	0.2441	0.2471	0.2411	0.2490
0.5	0.1438	0.1443	0.2118	0.1446
2	0.0719	0.0721	0.1565	0.0723
4	0.0610	0.0618	0.1319	0.0622
8	0.0513	0.0523	0.1099	0.0532

Here is an example showing the use of the tables. Consider a clamped–free beam whose width tapers linearly, with the width at the clamped end being twice the width at the free end. In Tables 7 and 13 with $a(l)/a(0) = 2$ and $n = 1$ the norm values are

$$\|\mathscr{F}\|_\infty = 0.3863, \quad \|\mathscr{H}\|_\infty = 0.5000.$$

The bounds for maximum deflection $\|y\|_\infty$ and maximum curvature $\|y''\|_\infty$ are given by formulae (4) and (6), which for this beam-column have the form

$$\|y\|_\infty \leqslant \frac{\|y_0\|_\infty}{1-0.3863|\lambda|}, \quad \|y''\|_\infty \leqslant \frac{\|y_0''\|_\infty}{1-0.5000|\lambda|},$$

where $\|y_0\|_\infty$ and $\|y_0''\|_\infty$ are the maximum deflection and curvature in the same beam with no axial load. The above bounds are valid for arbitrary transverse load conditions. To illustrate for a particular case: The maximum deflection in this tapered beam with a unit point transversal load at the free end and no axial load is given in Roark and Young (1975) as $\|y_0\|_\infty = 0.579l^3/(3EI)$. The maximum deflection for the same beam with axial load is therefore bounded by

$$\|y\|_\infty \leqslant \frac{0.579l^3/(3EI)}{1-0.3863|\lambda|}.$$

5. CONCLUSIONS

The formulae (4) and (5) have a form similar to Timoshenko's amplification factor. They have, however, the important property of being upper bounds. The formula has three parameters: the axial load magnitude λ; the maximum $\|y_0\|_\infty$ or $\|y_0''\|_\infty$, which in many cases may be found from standard beam tables; and the operator norm $\|\mathscr{F}\|_\infty$ or $\|\mathscr{H}\|_\infty$, which is fixed for a given beam-column geometry. Values of the norms for the most common beam geometries are tabulated in Section 4.

The physical interpretation of the deflection kernel provided by eqn (18) implies that it may be possible, at least in principle, to determine the kernel directly from experimental measurements. This idea has been proposed by Baruch (1973) and by Tai *et al.* (1982) for beam-column integral equations of different form than those considered here.

The proposed formula is different from the upper bound formula proposed by Flavin (1976). The proposed formula tends to be more accurate for small axial loads while Flavin's is more accurate for larger axial loads.

The present approach to deriving an upper bound formula lends itself more readily to generalization than Flavin's approach. In this paper upper bound formulae for curvature were derived, and similar formulae for slope, bending moment, or other functions may also be derived by the same procedure. Different operator norms, such as $\|y\|_2$, may also be bounded by formulae of the same form as those discussed here. Norm-based upper bound formulae may also be derived for more general beam-column problems, including varying axial load (for example the classic problem of the flagpole under its own weight) and nonideal boundary conditions (including for example finite stiffness terms). Norm-based upper bound formulae for plates and shells with axial stress may also be derived. These are topics for further study.

REFERENCES

Arbabi, F. and Li, F. (1991). Buckling of variable cross-section columns: Integral-equation approach. *J. Struct. Engng* **117**(8), 2426–2441.

Baruch, M. (1973). Integral equations for nondestructive determination of buckling loads for elastic plates and bars. *Israel J. Technol.* **11**(1–2), 1–8.

Flavin, J. N. (1976). A bound for the deflection of a beam-column. *Trans. ASME, J. Appl. Mech.* **43**(1), 181–182.

Horn, R. A. and Johnson, C. R. (1985). *Matrix Analysis*. Cambridge University Press, Cambridge.

Lee, S. Y. and Kuo, Y. H. (1991). Deflection and stability of elastically restrained nonuniform beam. *J. Engng Mech.* **117**(3), 674–692.
Mikhlin, S. G. (1964). *Integral Equations* (2nd Edn). Pergamon Press, Oxford.
Piché, R. (1993). Lower bounds for the critical buckling load using matrix norms. *J. Engng Mech.* **119**(5).
Roark, R. J. and Young, W. C. (1975). *Formulas for Stress and Strain* (5th Edn). McGraw-Hill, Maidenhead.
Tai, H., Katayama, T. and Sekiya, T. (1982). Buckling analysis by influence function. *Mech. Res. Comm.* **9**(3), 139–144.
Timoshenko, S. P. and Gere, J. M. (1961). *Theory of Elastic Stability* (2nd Edn). McGraw-Hill, Maidenhead.